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Inseparability of Quantum Parameters

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Abstract In this work, we show that 'splitting of quantum information' (Zhou, D., et al. in quant-ph/0503168) is an impossible task from three different but consistent principles namely unitarity of Quantum Mechanics, no-signaling condition and non-increase of entanglement under Local Operation and Classical Communication.

1 Introduction

In quantum information theory it is most important of knowing the various differences between the classical and quantum information. Many operations which are feasible in digitized information becomes an impossibility in quantum world [1–6]. This may be probably due to the linear structure or may be due to the unitary evolution in quantum mechanics. Regardless of their origin, these impossible operations are making quantum information processing much more restricted than it's classical counterpart. At the same time these restrictions on many quantum information processing tasks is making quantum information more secure. In the famous land mark paper of Wootters and Zurek it was shown that a single quantum cannot be cloned [1]. Later it was also shown by Pati and Braunstein that we cannot delete either of the two quantum states when we are provided with two identical quantum states at our input port [2]. In spite of these two famous 'no-cloning' [1] and 'no-deletion' [2] theorem there are many other 'no-go' theorems like 'no-self replication' [3], 'no-partial erasure' [5], 'no-splitting' [6] and many more which have come up. Recent

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research has revealed that these theorems are consistent with different principles like principle of no-signaling and conservation of entanglement under LOCC [7–9]. If we put it in a different way it means that if we violate these 'no-go' theorems we will violate the principle of no-signaling and non-increase of entanglement under LOCC.

No-Splitting Theorem It is a well known fact that there are many operations which are feasible in the classical world but doesn't hold good in the quantum domain. These we generally refer as 'General impossible operations'. "No splitting Theorem" [6] (almost equivalent to the 'No-Partial Erasure of Quantum Information' [5]) is yet another addition to this set. It states that 'For an unknown qubit,quantum information cannot be split into two complementing qubits, i.e. the information in one qubit is an inseparable entity. An important application of splitting of quantum information is a construction of a gate which can be used to reversibly split a parameter encoded in non-orthogonal quantum states, enabling the necessary quantum information compression and decompression required for optimal quantum cloning with multiple copies [10].

In this work our objective is different from [6] in the sense that here we will investigate whether we can split two non-orthogonal quantum states from three different but consistent principles like, preservation of inner products under unitary evolution, the principle of non-increase of entanglement under LOCC and principle of no signaling. In other words, unlike in Ref. [6], instead proving the splitting of quantum state from linearity, we correlate this impossibility with other aspects like, unitarity, restrictions on entanglement processing and causality.

2 Proof of No-Splitting Theorem from Unitarity of Quantum Mechanics

First of all we show that the 'No-splitting theorem' is consistent with the unitary evolution of quantum theory. For this purpose we will consider a pair of non-orthogonal states $[|\psi_1(\theta_1,\phi_1)\rangle, |\psi_1(\theta_2,\phi_2)\rangle]$, where $0\le\theta_1,\theta_2\le\pi$ and $0\le\phi_1,\phi_2\le 2\pi$. These non-orthogonal states are represented by points on the Bloch sphere. Let us assume that the splitting of quantum information into complementary parts is possible. If we consider a hypothetical machine which can split quantum information in each of the non-orthogonal states into complementary parts, then the action of the machine on the non-orthogonal states $[|\psi_1(\theta_1,\phi_1)\rangle, |\psi_1(\theta_2,\phi_2)\rangle]$ is defined by the set of transformations:

$$|\psi_1(\theta_1,\phi_1)\rangle|\psi_2\rangle \rightarrow |\psi_1(\theta_1)\rangle|\psi_2(\phi_1)\rangle, \tag{1}$$

$$|\psi_1(\theta_2,\phi_2)\rangle|\psi_2\rangle \rightarrow |\psi_1(\theta_2)\rangle|\psi_2(\phi_2)\rangle \tag{2}$$

where

$$|\psi_1(\theta_j,\phi_j)\rangle = \cos\left(\frac{\theta_j}{2}\right)|0\rangle + \sin\left(\frac{\theta_j}{2}\right)e^{(i\phi_j)}|1\rangle, \tag{3}$$

$$|\psi_2(\phi_j)\rangle = |\psi_{21}\rangle + e^{(i\phi_j)}|\psi_{22}\rangle, \tag{4}$$

$$|\psi_1(\theta_j)\rangle = \cos\left(\frac{\theta_j}{2}\right)|\psi_{11}\rangle + \sin\left(\frac{\theta_j}{2}\right)|\psi_{12}\rangle \tag{5}$$

where $j = (1, 2)$. Here $|\psi_{11}\rangle, |\psi_{12}\rangle, |\psi_{21}\rangle, |\psi_{22}\rangle$ are non-normalized states independent of θ, ϕ. The unitarity of transformations will preserve the inner product.

$$\langle\psi_1(\theta_1, \phi_1)|\psi_1(\theta_2, \phi_2)\rangle = \langle\psi_1(\theta_1)|\psi_1(\theta_2)\rangle\langle\psi_2(\phi_1)|\psi_2(\phi_2)\rangle. \tag{6}$$

The above equality will not hold for all values of (θ, ϕ). The equality will hold if $\phi_2 = \phi_1 + n\pi$ and $\theta_1 \pm \theta_2 = (2m + 1)\pi$ where m, n are integers (see Appendix). This equality corresponds to a situation where the quantum states are orthogonal. Thus we see that the equality does not hold for all values of θ and ϕ and hence we conclude that this kind of transformation doesn't exist. We cannot split the quantum information for two non-orthogonal quantum states.

3 From Non-Increase of Entanglement under LOCC and No Signaling Condition

Next we show that splitting of quantum information is an impossible operation from the principle of non-increase of entanglement under LOCC. Let us consider an entangled state shared by two distant parties Alice and Bob, of the form

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle_A|\psi_1(\theta_1, \phi_1)\rangle_B + |1\rangle_A|\psi_1(\theta_2, \phi_2)\rangle_B]|\psi_2\rangle_B \tag{7}$$

where $\{|\psi_2\rangle\}$ is the blank state attached to the Bob's particle.

The reduced density matrix on Alice's side is given by,

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(|\psi\rangle_{AB\,AB}\langle\psi|)\\ &= \frac{1}{2}[I + |1\rangle\langle 0|(\langle\psi_1(\theta_1, \phi_1)|\psi_1(\theta_2, \phi_2)\rangle)\\ &\quad + |0\rangle\langle 1|(\langle\psi_1(\theta_2, \phi_2)|\psi_1(\theta_1, \phi_1)\rangle)].\end{aligned} \tag{8}$$

Let us assume that Bob is in possession of a machine which will split the quantum information of his particle. The transformation describing the action of the machine is given by (1) and (2). Now after the application of the quantum information splitting machine the entangled state (7) takes the form

$$|\psi\rangle^C_{AB} = \frac{1}{\sqrt{2}}[|0\rangle_A|\psi_1(\theta_1)\rangle_B|\psi_2(\phi_1)\rangle_B + |1\rangle_A|\psi_1(\theta_2)\rangle_B|\psi_2(\phi_2)\rangle_B]. \tag{9}$$

The reduced density matrix on the Alice's side after the application of the machine is given by,

$$\begin{aligned}\rho^C_A = \frac{1}{2}[I &+ |1\rangle\langle 0|(\langle\psi_1(\theta_1)|\psi_1(\theta_2)\rangle)(\langle\psi_2(\phi_1)|\psi_2(\phi_2)\rangle)\\ &+ |0\rangle\langle 1|(\langle\psi_1(\theta_2)|\psi_1(\theta_1)\rangle)(\langle\psi_2(\phi_2)|\psi_2(\phi_1)\rangle)].\end{aligned} \tag{10}$$

The respective largest eigenvalues of these two reduced density matrices are given by

$$\lambda_A = \frac{1}{2} + \frac{|p|^2}{2}, \tag{11}$$

$$\lambda^C_A = \frac{1}{2} + \frac{|q|^2|r|^2}{2} \tag{12}$$

where $p = \langle\psi_1(\theta_2,\phi_2)|\psi_1(\theta_1,\phi_1)\rangle$, $q = \langle\psi_1(\theta_2)|\psi_1(\theta_1)\rangle$, $r = \langle\psi_2(\phi_2)|\psi_2(\phi_1)\rangle$. To show that the amount of entanglement $E(|\psi\rangle_{AB})$ and $E(|\psi\rangle_{AB}^C)$ of the respective entangled states before and after the splitting doesn't increase, we must show that $\lambda_A < \lambda_A^C$. To show $\lambda_A < \lambda_A^C$ this we must show that,

$$|\langle\psi_1(\theta_2,\phi_2)|\psi_1(\theta_1,\phi_1)\rangle| < |\langle\psi_1(\theta_2)|\psi_1(\theta_1)\rangle||\langle\psi_2(\phi_2)|\psi_2(\phi_1)\rangle|, \tag{13}$$

$$\begin{aligned}
&\text{LHS:}\quad \left|\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + e^{i(\phi_1-\phi_2)}\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right|,\\
&\text{RHS:}\quad \left|\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\right|,\\
&|1+e^{i(\phi_1-\phi_2)}|.
\end{aligned} \tag{14}$$

Let

$$(\phi_1-\phi_2) = k, \qquad \cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) = x, \tag{15}$$

$$\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) = y \tag{16}$$

where x and y are real quantities.

Now

$$\begin{aligned}
|\langle\psi_1(\theta_2,\phi_2)|\psi_1(\theta_1,\phi_1)\rangle| &= |x+e^{ik}y| = |[x+y\cos(k)]+iy[\sin(k)]|\\
&= \sqrt{[x^2+y^2+2xy\cos(k)]}
\end{aligned}$$

and

$$|\langle\psi_1(\theta_2)|\psi_1(\theta_1)\rangle||\langle\psi_2(\phi_2)|\psi_2(\phi_1)\rangle| = (x+y)\sqrt{2(1+\cos(k))}.$$

Therefore

$$\begin{aligned}
A(\text{say}) &= [|\langle\psi_1(\theta_2,\phi_2)|\psi_1(\theta_1,\phi_1)\rangle|]^2 - [|\langle\psi_1(\theta_2)|\psi_1(\theta_1)\rangle||\langle\psi_2(\phi_2)|\psi_2(\phi_1)\rangle|]^2\\
&= x^2+y^2+2xy\cos(k) - 2(x+y)^2(1+\cos(k))\\
&= -[x^2+y^2+4xy+2(x^2+y^2+xy)\cos(k)] > 0
\end{aligned}$$

for some values of x, y, k. This implies that, $\lambda_A > \lambda_A^C \Rightarrow E(|\psi\rangle_{AB}^C) > E(|\psi\rangle_{AB})$, as a consequence of which we can say that the amount of entanglement will increase under local operation. However we know that entanglement is non-increasing under such operations (in general one can only claim that it is conserved under a bilocal unitary operation). This gives rise to contradiction. Therefore, it is clear that the principle of non-increase of entanglement under LOCC doesn't allow perfect splitting of non-orthogonal quantum states. This rules out the existence of a hypothetical quantum information splitting machine, designed to split the quantum information of a non-orthogonal quantum state.

Next we show that the splitting of quantum information is not possible from the principle of no signaling. In other words we can say that if we assume perfect splitting of quantum information it will violate the principle of no-signaling.

Suppose we have a singlet state shared by two distant parties Alice and Bob. The singlet state can be written in two different basis as

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle|\overline{\psi_1}\rangle - |\overline{\psi_1}\rangle|\psi_1\rangle) = \frac{1}{\sqrt{2}}(|\psi_2\rangle|\overline{\psi_2}\rangle - |\overline{\psi_2}\rangle|\psi_2\rangle) \tag{17}$$

where $\{|\psi_1\rangle, |\overline{\psi_1}\rangle\}$ and $\{|\psi_2\rangle, |\overline{\psi_2}\rangle\}$ are two sets of mutually orthogonal spin states (qubit basis). Alice possesses the first particle while Bob possesses the second particle. Alice can choose to measure the spin in any one of the qubit basis namely $\{|\psi_1\rangle, |\overline{\psi_1}\rangle\}$, $\{|\psi_2\rangle, |\overline{\psi_2}\rangle\}$. The theorem of no signaling tells us that the measurement outcome of Bob are invariant under local unitary transformation done by Alice on her qubit. The density matrix $\rho_B = \mathrm{tr}\,\rho_{AB} = \mathrm{tr}[(U_A \otimes I_B)\rho_{AB}(U_A \otimes I_B)^\dagger]$ is invariant under local unitary operation by Alice. Hence Bob cannot distinguish two mixtures due to the unitary operation done at remote place. One may ask if Bob split the quantum information of his particle and if Alice measure her particle in either of the two basis then is there any possibility that Bob know the basis in which Alice measure her qubit or in other words, is there any way by which Bob using a perfect splitting machine can distinguish the statistical mixture in his subsystem resulting from the measurement done by Alice. If Bob can do this then signaling will take place, which is impossible. Hence now our task is to show that the splitting of information is an impossible task from no-signaling principle.

Let us consider a situation where Bob is in possession of a hypothetical quantum information splitting machine. The unitary transformation describing the splitting of quantum information for an input state $|\psi_i(\theta, \phi)\rangle$ (where $i = 1, 2$) is defined as

$$\begin{aligned} &|\psi_i(\theta, \phi)\rangle|\Sigma\rangle \rightarrow |\psi_i(\theta)\rangle|\Sigma(\phi)\rangle, \\ &|\overline{\psi_i(\theta, \phi)}\rangle|\Sigma\rangle \rightarrow |\overline{\psi_i(\theta)}\rangle|\overline{\Sigma(\phi)}\rangle \end{aligned} \tag{18}$$

where $\{|\Sigma\rangle\}$ is the ancilla state attached by Bob.

After the application of the transformation defined in (18) by Bob on his particle the singlet state defined by (17) including the ancilla state attached by Bob reduces to the form,

$$\begin{aligned} |\chi\rangle|\Sigma\rangle \rightarrow |\chi\rangle^S &= \frac{1}{\sqrt{2}}[|\psi_1(\theta, \phi)\rangle|\overline{\psi_1(\theta)}\rangle|\overline{\Sigma(\phi)}\rangle - |\overline{\psi_1(\theta, \phi)}\rangle|\psi_1(\theta)\rangle|\Sigma(\phi)\rangle] \\ &= \frac{1}{\sqrt{2}}[|\psi_2(\theta, \phi)\rangle|\overline{\psi_2(\theta)}\rangle|\overline{\Sigma(\phi)}\rangle - |\overline{\psi_2(\theta, \phi)}\rangle|\psi_2(\theta)\rangle|\Sigma(\phi)\rangle]. \end{aligned} \tag{19}$$

After Bob applying the splitting machine on his qubit, Alice can measure her particle in two different basis. If Alice measures her particle in the basis $\{|\psi_1\rangle, |\overline{\psi_1}\rangle\}$, then the reduced density matrix in the Bob's subsystem (including ancilla) is given by,

$$\begin{aligned} \rho_{BC} &= \mathrm{tr}_A(\rho_{ABC}) \\ &= \frac{1}{2}\{|\overline{\psi_1(\theta)\Sigma(\phi)}\rangle\langle\overline{\psi_1(\theta)\Sigma(\phi)}| + |\psi_1(\theta)\Sigma(\phi)\rangle\langle\psi_1(\theta)\Sigma(\phi)|\}. \end{aligned} \tag{20}$$

On the other hand if Alice measures her particle in the basis $\{|\psi_2\rangle, |\overline{\psi_2}\rangle\}$ then the state described by the reduced density matrix in the Bob's side is given by,

$$\begin{aligned} \rho_{BC} &= \mathrm{tr}_A(\rho_{ABC}) \\ &= \frac{1}{2}\{|\overline{\psi_2(\theta)\Sigma(\phi)}\rangle\langle\overline{\psi_2(\theta)\Sigma(\phi)}| + |\psi_2(\theta)\Sigma(\phi)\rangle\langle\psi_2(\theta)\Sigma(\phi)|\}. \end{aligned} \tag{21}$$

Since the statistical mixture in (20) and (21) are different, so this would have allowed Bob to distinguish in which basis Alice has performed measurement, thus allowing for super luminal signaling. However the criterion of 'No-signaling' tells us that communication faster than light is not possible. So we arrive at a contradiction, that is, the transformation defined in (18) is not possible in quantum world. This rules out the existence of hypothetical machine like quantum information splitting machine.

4 Conclusion

In this work we show that the total information contained in the quantum state on the Bloch-sphere cannot be written as the tensor product of the state containing the information of the azimuthal angle and the state containing the information of the phase angle. Therefore the quantum information can be regarded as an inseparable entity or in other words we can say that it is impossible to express the function of θ and ϕ as the product of function of θ alone and function of ϕ alone. To justify the above statement, we proved the no-splitting theorem from three different principles: (i) Unitarity of quantum mechanics (ii) principles of non-increase of entanglement under LOCC and (iii) Principle of no signaling.

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Appendix

$\langle\psi_1(\theta_1,\phi_1)|\psi_1(\theta_2,\phi_2)\rangle = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\phi_2-\phi_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$, $\langle\psi_1(\theta_1)|\psi_1(\theta_2)\rangle = \cos\frac{\theta_1}{2}\times\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$, and $\langle\psi_2(\phi_1)|\psi_2(\phi_2)\rangle = 1 + e^{i(\phi_2-\phi_1)}$. Now we have to show that, $\langle\psi_1(\theta_1,\phi_1)|\psi_1(\theta_2,\phi_2)\rangle = \langle\psi_1(\theta_1)|\psi_1(\theta_2)\rangle\langle\psi_2(\phi_1)|\psi_2(\phi_2)\rangle$. This equality is possible only when

$$\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = -e^{i(\phi_2-\phi_1)}. \tag{22}$$

Now by equating the real and imaginary parts of the above expression we get,

$$\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \cos(\phi_2-\phi_1), \tag{23}$$

$$\sin(\phi_2-\phi_1) = 0. \tag{24}$$

On simplifying equation (19) we get, $(\phi_2-\phi_1) = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. Using $(\phi_2-\phi_1) = n\pi$ equation (17) reduces to the form $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = (-1)^{n+1}$.

Now we consider two cases:

Case 1. When n is even, $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = -1 \Rightarrow (\theta_1-\theta_2) = (2m+1)\pi$.

Case 2. When n is odd, $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = 1 \Rightarrow (\theta_1+\theta_2) = (2m+1)\pi$ (where, $m = 0, \pm 1, \pm 2, \ldots$). Thus we see that the unitarity of the transformation is preserved only when $\phi_2 = \phi_1 + n\pi$ and $\theta_1 \pm \theta_2 = (2m+1)\pi$.

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